

THE PROBLEM OF SYMMETRY IN THE THEORY OF ELASTICITY PROBLEM CONCERNING A PLANAR TENSILE CRACK†

A. G. RAMM and Ye. I. SHIFRIN

Manhattan, Kansas (USA) and Moscow

e-mail: ramm@math.ksu.edu

shifrin@ipmnet.ru

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The problem of a bounded planar crack in a homogeneous, isotropic, linearly elastic space is considered. It is assumed that uniform normal loads, which are equal in magnitude and opposite in direction, are applied to the crack surfaces. It is shown that a circle is the only form of crack for which the stress intensity factor is constant along its contour. © 2005 Elsevier Ltd. All rights reserved.

1. FORMULATION OF THE PROBLEM

Suppose a crack occupies a bounded domain G in the plane $x_3 = 0$ of an unbounded, homogeneous and isotropic elastic space. Normal loads, which are equal in magnitude and opposite in direction, are applied to the crack surfaces and the displacements tend to zero at infinity. We will denote the load applied to the upper surface of the crack by $t(x_1, x_2)$. It is well known that the problem reduces to a pseudodifferential equation in the crack domain [1]

$$p_G \Lambda u(x_1, x_2) = \frac{1-\nu}{\mu} t(x_1, x_2), \quad (x_1, x_2) \in G; \quad u(x_1, x_2) = 0, \quad (x_1, x_2) \notin G \quad (1.1)$$

Here, $u(x_1, x_2)$ is the displacement of the upper surface of the crack, μ is the shear modulus, ν is Poisson's ratio and Λ is a pseudodifferential operator with a symbol $|\xi| = \sqrt{\xi_1^2 + \xi_2^2}$, $\xi = (\xi_1, \xi_2)$

$$\Lambda u = F^{-1}(|\xi| \tilde{u}(\xi)), \quad \tilde{u}(\xi) = \int_{R^2} u(x_1, x_2) \exp[i(x_1 \xi_1 + x_2 \xi_2)] dx_1 dx_2$$

where $\tilde{u}(\xi)$ is the Fourier transform of the function $u(x_1, x_2)$, F^{-1} is the inverse Fourier transform and p_G is the restriction to the crack domain G .

Despite the fact that the operator Λ is not local and, in the case of this operator, there is no corresponding maximum principle, it was found that the solution of Eq. (1.1) possess many of the properties of the Dirichlet problem for Poisson's equation. In particular, comparison theorems [2–4] and isoperimetric inequalities [5–7], which are analogous to those which hold in the case of Poisson's equation (also, see [8, 9]) were proved for the solutions of problem (1.1). The existence of a relation between the solutions of Eq. (1.1) and the solutions of Poisson's equation has also been pointed out [10].

The aim of this paper is to transfer a further result, which holds for Poisson's equation, to Eq. (1.1). The following assertion has been proved by Aleksandrov's moving hyperplane method [11].

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Suppose $u(x)$ is the solution of the Dirichlet problem for Poisson's equation

$$\Delta u(x) = -1, \quad x \in D \subset R^n; \quad u(x') = 0, \quad x' \in \partial D \quad (1.2)$$

Here D is a bounded domain in R^n with a fairly smooth boundary ∂D . If the condition

$$\partial u(x')/\partial n = \text{const} \quad (1.3)$$

is satisfied, where \mathbf{n} is a unit outward normal to ∂D , then the domain D is a sphere.

Following the publication of [11], several further papers appeared in which this result was proved by other methods [12, 13].

Now, suppose $u(x_1, x_2)$ is the solution of Eq. (1.1). It is well known that, close to the smooth boundary of the crack ∂G , the function $u(x_1, x_2)$ has the following asymptotic form

$$u(x_1, x_2) \approx 2 \frac{1 - \nu}{\mu} \frac{K_I(x'_1, x'_2)}{\sqrt{2\pi}} s^{1/2}, \quad (x'_1, x'_2) \in \partial G \quad (1.4)$$

Then, (x_1, x_2) lies on the normal to the boundary ∂G which passes through the point (x'_1, x'_2) , s is the distance between the points (x_1, x_2) and (x'_1, x'_2) and $K_I(x'_1, x'_2)$ is the stress intensity factor (SIF).

We will assume that the loads applied to the crack surfaces, are uniform, that is,

$$t(x) = t_0 = \text{const} > 0 \quad (1.5)$$

It is seen from relation (1.4) that, in the tensile crack problem, the condition

$$K_I(x'_1, x'_2) = K_I^0 = \text{const} \quad (1.6)$$

is an analogue of condition (1.3).

Consequently, in the case of Eq. (1.1), the question can be formulated in the following manner: is it true that, if in the case of a uniform normal load the stress intensity factor is constant along the contour of a bounded crack, the crack has the form of a circle?

We also note that the problem of constructing the crack domains, for which the stress intensity factor is constant along the contour for different types of specified loads, has been discussed in [8]† in connection with the construction of the extremal contours of cracks.

2. FORMULATION AND SCHEME FOR THE PROOF OF THE THEOREM

Theorem 1. Suppose uniform normal loads $t(x_1, x_2) = t_0$ are applied to the surfaces of a crack G , which is bounded by a simple, closed, smooth (class C^∞) contour ∂G and that the stress intensity factor is constant along the crack contour ($K_I(x'_1, x'_2) = K_I^0$). Then, the crack domain G is a circle.

To prove this theorem, following a well-known approach [11], we will use Aleksandrov's moving hyperplane method. This method is based on several geometrical assertions. We will present their two-dimensional formulations, which will be required later.

Assertion 1. Suppose G is a bounded planar domain and a straight line is found for any direction which has the given direction and is such that the domain G is symmetrical about this line. Then, G is a circle.

We will assume that ∂G is a simple, closed, smooth curve. We take an arbitrary direction and straight line T which has this direction and does not intersect the domain G . We start to move this line in the direction of the domain G such that it remains parallel to itself. The line T divides the plane into two half-planes R_-^2 and R_+^2 which are located in directions opposite to and the same as the direction of motion of the line T , respectively. As the line T advances, it initially comes into contact with the closure of domain G , which is denoted by \bar{G} and then starts to intersect it. In this case, the domain G is divided into two parts G_- and G_+ belonging to the half-planes R_-^2 and R_+^2 . We now reflect the domain G_- symmetrically with respect to the line T and we denote the image of G_- , under this mapping, by G_s .

†See also: GOLDSHTEIN, R. V. and ENTOV, V. M., Some Qualitative Methods in Fracture Mechanics. Preprint No. 76, Inst. Problem Mekh. Akad. Nauk SSSR, Moscow, 1976.

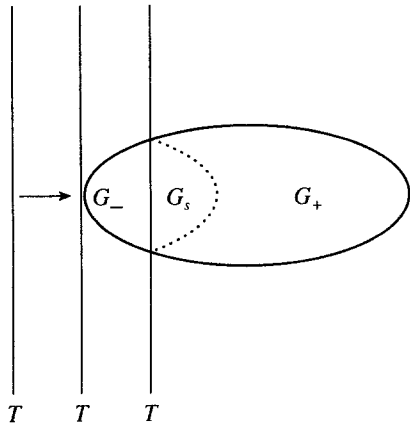


Fig. 1

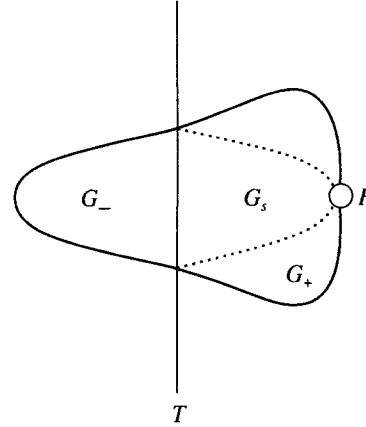


Fig. 2

(Fig. 1). As the line T advances, the domain G_- increases and the domain G_+ decreases. By virtue of this, the domain G_s will initially belong to G_+ and subsequently cease to belong to this domain. We consider the limiting position of the line T for which the domain G_s still belongs to the domain G_+ .

Assertion 2. Only two versions of the limiting position are possible:

- (1) the boundary of the domain G_s touches the boundary of the domain G_+ at a certain point $P \in \partial G$ (Fig. 2);
- (2) the line T becomes orthogonal to the boundary ∂G at a certain point Q of the intersection of T with ∂G (Fig. 3).

It will be shown below that the assumption that $G_s \neq G_+$ in the case of the limiting position of the line T leads to a contradiction. Consequently, the domains G_s and G_+ are identical in the case of the limiting position of the line T , which means that the domain G is symmetrical with respect to T .

Since this assertion will be proved for a limiting line T in an arbitrary direction, it follows from assertion 1 that G is a circle.

3. PROOF OF THE AUXILIARY ASSERTIONS AND THEOREMS

For the proof, it will be more convenient to use the equation of the tensile crack problem in the form, which differs from (1.1) but is equivalent to it. This problem can be reduced to a mixed boundary-value problem for a function which is harmonic in the half-space (for example, see [2]).

We consider the half-space $R_+^3 = \{\mathbf{x} = (x_1, x_2, x_3): x_3 > 0\}$.

We will assume that the domain G lies in the plane $x_3 = 0$. Suppose $U(\mathbf{x})$ is a function harmonic in R_+^3 . We specify the following mixed conditions on the boundary $x_3 = 0$

$$U(x_1, x_2, 0) = 0, \quad (x_1, x_2) \notin G \tag{3.1}$$

$$\frac{\partial U(x_1, x_2, 0)}{\partial x_3} = -\frac{1-\nu}{\mu} t_0, \quad (x_1, x_2) \in G, \quad t_0 > 0 \tag{3.2}$$

We will also assume that

$$U(\mathbf{x}) \rightarrow 0 \quad \text{when} \quad |\mathbf{x}| \rightarrow \infty$$

We will use the notation $U(x_1, x_2, 0) = u(x_1, x_2)$. Thus, the function $u(x_1, x_2)$ which has been defined satisfies Eq. (1.1) with $t(x_1, x_2) = t_0$. In turn, if $u(x_1, x_2)$ satisfies Eq. (1.1), the function $U(\mathbf{x})$, which is harmonic in the half-space and equal to $u(x_1, x_2)$ when $x_3 = 0$, satisfies boundary conditions (3.1) and (3.2).

We now take an arbitrary plane Π which is parallel to the x_3 axis and does not intersect G . The line of intersection of the plane Π and $x_3 = 0$ is denoted by T . We now begin to move the plane Π parallel to itself in the direction domain G , during which the line T initially touches the domain G and subsequently starts to intersect it. The plane Π separates the half-space R_+^3 into two quarter-spaces.

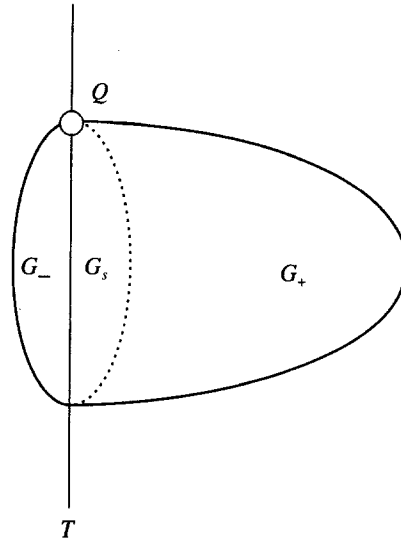


Fig. 3

One of these, which lies, relative to Π , in the direction opposite to the direction of motion, is denoted by S_- . The other quarter-space is denoted by S_+ . As above, we shall denote the part of the domain G belonging to S_- by G_- and the other part of the domain G , which belongs to S_+ , by G_+ . We now reflect S_- into S_+ symmetrically with respect to the plane Π , during which the domain G_- is symmetrically reflected about the line T into the domain G_s , which lies on the boundary of the half-space R_+^3 and belongs to S_+ . As has already been noted above, the domain G_s will initially belong to G_+ and subsequently ceases to belong to this domain. We now consider the limiting position of the plane Π for which the domain G_s still belongs to the domain G_+ . We introduce a system of coordinates such that the limiting position of the plane Π coincides with the plane $x_1 = 0$. In this case, $S_+ = \{x: x_3 \geq 0 \text{ and } x_1 \geq 0\}$. We construct the function in S_+

$$V(x_1, x_2, x_3) = U(-x_1, x_2, x_3) \tag{3.3}$$

It follows from definition (3.3) and the fact that the function $U(x)$ is harmonic that $V(x)$ is a function which is harmonic in S_+ . The boundary $S_+(\partial S_+)$ consists of two half-planes: $\Pi_+ = \Pi \cap R_+^3$ and $F_+ = \{(x_1, x_2, 0), x_1 \geq 0\}$.

It follows from definition (3.3) that

$$V(x) = U(x) = U(0, x_2, x_3), \quad x \in \Pi_+ \tag{3.4}$$

According to relations (3.1)–(3.3), on the boundary F_+ we have

$$V(x_1, x_2, 0) = 0, \quad (x_1, x_2) \notin G_s \tag{3.5}$$

$$\frac{\partial V(x_1, x_2, 0)}{\partial x_3} = \frac{\partial U(-x_1, x_2, 0)}{\partial x_3} = -\frac{1-\nu}{\mu} t_0, \quad (x_1, x_2) \in G_s \tag{3.6}$$

Since, later, we shall repeatedly making use of the enhanced Hopf maximum principle [14], we present it here in the form in which it will subsequently be required.

The enhanced Hopf maximum principle. Suppose the function $W(x)$ is defined and continuous in the closure \bar{D} of the domain $D \subset R^n$ and, within D , it satisfies the inequality

$$\Delta W(x) \leq 0$$

Then, $W(x)$ reaches a minimum value on the boundary ∂D . We will assume that $W(x)$ reaches a minimum value at the point $x'_0 \in \partial D$ and that it can be inscribed in the D sphere, which intersects with ∂D at a

single point x'_0 . In particular, this is possible if the surface ∂D is smooth in a certain neighbourhood of x'_0 . We also assume that, at the point x'_0 , there is a derivative of the function $W(x)$ which respect to a certain direction l which is not tangential to ∂D at the given point and which goes inside D , then either $\partial W(x'_0)/\partial l > 0$ or $W(x) \equiv W(x'_0) = \text{const}$.

Lemma 1. The inequality $U(\mathbf{x}) \geq V(\mathbf{x})$ holds for $\mathbf{x} \in S_+$ and, moreover, if $G_s \neq G_+$, then $U(\mathbf{x}) > V(\mathbf{x})$ within S_+ .

Proof. It follows from the comparison theorems [2, 4] that the solution of Eq. (1.1) in the case when $t(x_1, x_2) \geq 0$ is non-negative and, if $t(x_1, x_2) \neq 0$, then $u(x_1, x_2) > 0$ within G . Consequently, on the boundary R_+^3 , the function $U(\mathbf{x})$ is non-negative, and it follows from the maximum principle that $U(\mathbf{x}) \geq 0$ in R_+^3 and, if $u(x_1, x_2) \neq 0$, then $U(\mathbf{x}) > 0$ when $x_3 > 0$.

We now consider the function

$$W(\mathbf{x}) = U(\mathbf{x}) - V(\mathbf{x}) \quad (3.7)$$

in the quarter space S_+ .

It follows from the fact that $U(\mathbf{x})$ and $V(\mathbf{x})$ are harmonic functions that $W(\mathbf{x})$ is a function which is harmonic in S_+ .

It follows from definition (3.7) and equalities (3.4) that

$$W(\mathbf{x}) = W(0, x_2, x_3) = 0, \quad \mathbf{x} = (0, x_2, x_3) \in \Pi_+ \quad (3.8)$$

From conditions (3.1) and (3.5), we have

$$W(x_1, x_2, 0) = 0, \quad (x_1, x_2, 0) \notin G_+ \quad (3.9)$$

From the fact that $u(x_1, x_2) > 0$ in G_+ and condition (3.5), we have

$$W(x_1, x_2, 0) > 0, \quad (x_1, x_2, 0) \in G_+ \setminus G_s \quad (3.10)$$

From the fact that the function $U(\mathbf{x})$ tends to zero at infinity, we obtain

$$W(\mathbf{x}) \rightarrow 0 \quad \text{when} \quad |\mathbf{x}| \rightarrow \infty \quad (3.11)$$

Hence, according to relations (3.8)–(3.11), if there is a point on the boundary ∂S_+ at which $W(\mathbf{x}) < 0$, then this point belongs to the domain G_s . In this case, the function $W(\mathbf{x})$ has a negative minimum which is reached in this domain. However, according to conditions (3.2) and (3.6),

$$\frac{\partial W(x_1, x_2, 0)}{\partial x_3} = \frac{\partial U(x_1, x_2, 0)}{\partial x_3} - \frac{\partial V(x_1, x_2, 0)}{\partial x_3} = 0 \quad \text{when} \quad (x_1, x_2, 0) \in G_s \quad (3.12)$$

It follows from this and from the enhanced Hopf maximum principle that $W(\mathbf{x}) \equiv \text{const}$, which contradicts the assumption. Hence, $W(\mathbf{x}) \geq 0$ on ∂S_+ and $W(\mathbf{x}) > 0$ within S_+ .

Lemma 2. Suppose stress intensity factors, which are defined by a function which is harmonic in a half-space and satisfies boundary conditions (3.1) and (3.2), are constant along the contour ∂G , $K_I(x'_1, x'_2) = K_I^0$. Then, if $G_s \neq G_+$, version 1 of the limiting position cannot be realized.

Proof. Consider the function $W(\mathbf{x})$ in S_+ . It follows from Lemma 1 that $W(\mathbf{x})$ is non-negative and takes a minimum value which is equal to zero on ∂S_+ . In particular, $W(\mathbf{x}) = 0$ at the point P (Fig. 2). In the neighbourhood of the point P , we introduce a local curvilinear system of coordinates (s, y_2, y_3) , where s is the natural parameter of a curve measured from point P , the y_2 axis is directed out of the domain G along the principal normal to ∂G at a point of the boundary with a parameter s and we leave the y_3 axis codirected with the x_3 axis but having its origin at the current point with the parameter s . In the $y_2 y_3$ plane, we change to the system of polar coordinates

$$y_2 = r \cos \theta, \quad y_3 = r \sin \theta, \quad 0 \leq \theta \leq \pi$$

The asymptotic form of a function, which is harmonic in a half-space and satisfies boundary conditions (3.1) and (3.2) close to the boundary ∂G , can be obtained from the results of an analysis of the behaviour of the solutions of elliptic equations in the neighbourhood of an edge [15]. In the system of coordinates (s, r, θ) , on taking account of the condition $K_I = K_I^0$, we have [16, 17]

$$U(\mathbf{x}) = \frac{1-\nu}{\mu} \left\{ \frac{\sqrt{2}}{\sqrt{\pi}} K_I^0 r^{1/2} \sin \frac{\theta}{2} - t_0 r \sin \theta \right\} + O(r^{3/2}) \quad (3.13)$$

In particular, expansion (3.13) holds in a plane which is orthogonal to the boundary ∂G at the point P . It follows from the definition of the function $V(\mathbf{x})$ that, in this plane, it has an asymptotic form which are identical with (3.13). From this, we obtain the asymptotic form of the function $W(\mathbf{x})$ in this plane in the neighbourhood of the point P

$$W(\mathbf{x}) = U(\mathbf{x}) - V(\mathbf{x}) = O(r^{3/2})$$

It can be seen that, in a plane which is orthogonal to the boundary ∂G at the point P , there are derivatives of the function $W(\mathbf{x})$ along directions which do not lie in the plane of the crack and, moreover, these derivatives are equal to zero. In particular, $\partial W(P)/\partial x_3 = 0$. It follows from this and from the enhanced Hopf maximum principle that $W(\mathbf{x}) \equiv \text{const}$, which contradicts the assumption that $G_s \neq G_+$.

Lemma 3. Suppose the stress intensity factors, which are defined by a function which is harmonic in the half-space and satisfies boundary conditions (3.1) and (3.2), are constant along the contour ∂G , $K_I(x'_1, x'_2) = K_I^0$. Then, if $G_s \neq G_+$, version 2 of the limiting position cannot be realized.

The proof rests on the following auxiliary assertion.

Lemma 4. Suppose that, in the case of a function which is harmonic in a half-space, boundary conditions (3.1) and (3.2) are satisfied, $K_I(x'_1, x'_2) = K_I^0$, version 2 of the limiting position is realized and $G_s \neq G_+$. Consider the point Q (Fig. 3). At this point $W(Q) = 0$. We take an arbitrary direction l which is not tangential to ∂S_+ at the point Q and goes inside the domain S_+ . It follows from the above conditions that the function $W(\mathbf{x})$ has a first and second derivative in the direction l at the point Q and, moreover, these derivatives are equal to zero

$$W_l(Q) = 0, \quad W_{ll}(Q) = 0 \quad (3.14)$$

Proof. In the (s, r, θ) coordinates introduced above in the neighbourhood of the point Q , the asymptotic expansion of the function $U(\mathbf{x})$

$$U(\mathbf{x}) = \frac{1-\nu}{\mu} \left\{ \frac{\sqrt{2}}{\sqrt{\pi}} K_I^0 r^{1/2} \sin \frac{\theta}{2} + \frac{r^{3/2}}{3} \left(M(s) \sin \frac{3\theta}{2} + K_I^0 \kappa(s) \sin \frac{3\theta}{2} \right) \right\} - t_0 r \sin \theta + O(r^{5/2}) \quad (3.15)$$

which is harmonic in the half-space and satisfies conditions (3.1) and (3.2), holds. Here, $M(s)$ is a smooth function defined on the contour and $\kappa(s)$ is the curvature of the contour.

The smooth part of the function, which is quadratic in the variables x_1, x_2, x_3 , is missing from (3.15) by virtue of boundary conditions (3.1) and (3.2).

We introduce the rectangular system of coordinates $Qz_1z_2z_3$ in the neighbourhood of the point Q such that the z_1 axis is directed along the tangent to ∂G at the point Q , the z_2 axis is directed along the principal normal outside the domain G and the z_3 axis is parallel to the x_3 axis. We now change to the spherical coordinates

$$z_1 = \rho \sin \psi \cos \varphi, \quad z_2 = \rho \sin \psi \sin \varphi, \quad z_3 = \rho \cos \psi; \quad 0 < \psi < \pi/2, \quad 0 < \varphi < \pi$$

We consider a half line which is defined by the angles $\psi = \psi_0$, $\varphi = \varphi_0$. We write the expansion (3.15), which holds in a certain neighbourhood of Q , for a point of this half-line and express the variables r and θ appearing in (3.15) in terms of the variable ρ and the angles ψ_0 and φ_0 . For this purpose, we take a point on the half-time A^0 with coordinates $(\rho, \varphi_0, \psi_0)$, which correspond to the Cartesian coordinates (z_1^0, z_2^0, z_3^0) .

Suppose that, in the neighbourhood of the point Q the equation of ∂G in the plane $z_3 = 0$ has the form $z_2 = \Phi(z_1)$. On taking account of the fact that the z_1 axis is a tangent to this curve, the function $\Phi(z_1)$ can be written in the form

$$\Phi(z_1) = -\kappa_Q z_1^2/2 + O(z_1^3) \quad (3.16)$$

where κ_Q is the curvature of the contour at the point Q .

The equation of the plane which is normal to ∂G and passes through the point $A^* = (z_1^*, \Phi(z_1^*), 0)$, has the form

$$z_1 - z_1^* + \Phi'(z_1^*)(z_2 - \Phi(z_1^*)) = 0 \quad (3.17)$$

Suppose this plane intersects the half-line being considered at the point A^0 . We now express z_1^* in terms of ρ . In order to do this, we substitute

$$z_1 = z_1^0 = \rho \sin \psi_0 \cos \varphi_0, \quad z_2 = z_2^0 = \rho \sin \psi_0 \sin \varphi_0$$

and expression (3.16) when $z_1 = z_1^*$ into Eq. (3.17). On solving the resulting equation for z_1^* , we obtain z_1^* and then also $z_2^* = \Phi(z_1^*)$.

We now calculate the distance r between the points A^0 and A^* and obtain

$$r = \rho q_0^{1/2} \left[1 + \kappa_Q \rho \frac{\sin^3 \psi_0 \sin \varphi_0 \cos^2 \varphi_0}{2q_0} + O(\rho^2) \right], \quad q_0 = \sin^2 \psi_0 \sin^2 \varphi_0 + \cos^2 \psi_0 \quad (3.18)$$

We next determine the magnitude of the angle θ . Using the equality

$$\rho \cos \psi_0 = r \sin \theta \quad (3.19)$$

and relation (3.18), we obtain

$$\theta = \theta_0 - \kappa_Q \rho \frac{\sin^2 \psi_0 \cos \psi_0 \cos^2 \varphi_0}{2q_0} + O(\rho^2), \quad \theta_0 = \arcsin \frac{\cos \psi_0}{q_0^{1/2}} \quad (3.20)$$

Substituting expressions (3.18)–(3.20) into the asymptotic expansion (3.15), we obtain

$$\begin{aligned} U(\mathbf{x}) = & \frac{1-\nu}{\mu} \left\{ \sqrt{\frac{2}{\pi}} \left[K_I^0 \rho^{1/2} q_0^{1/4} \sin \frac{\theta_0}{2} + K_I^0 \kappa_Q \rho^{3/2} \frac{\sin^2 \psi_0 \cos^2 \varphi_0}{4q_0^{3/4}} \times \right. \right. \\ & \times \left(\sin \frac{\theta_0}{2} \sin \psi_0 \sin \varphi_0 - \cos \frac{\theta_0}{2} \cos \psi_0 \right) + \\ & \left. \left. + \frac{\rho^{3/2} q_0^{3/4}}{3} \left(M(0) \sin \frac{3\theta_0}{2} + K_I^0 \kappa_Q \sin^3 \frac{\theta_0}{2} \right) \right] - t_0 \rho \cos \psi_0 \right\} + O(\rho^{5/2}) \end{aligned} \quad (3.21)$$

We now consider the half-line defined by the angles ψ_0 and φ_0 and which lies in the domain S_+ ($0 < \psi_0 < \pi/2$, $0 < \varphi_0 < \pi/2$). Under a reflection of S_- into S_+ , which is symmetrical about the $z_1 = 0$ plane, the half-line defined by the angles ψ_0 and φ_0^* , where $\varphi_0^* = \pi - \varphi_0$, maps into the indicated half-line and, moreover, the points $(\rho, \varphi_0^*, \psi_0)$ map into the points $(\rho, \varphi_0, \psi_0)$. It follows from expression (3.21), the definition of q_0 (the second expression of (3.18)) and the definition of θ (the second expression of (3.20)) that the terms in the expansion of the function $U(\mathbf{x})$ with an order of smallness with respect to ρ of lower than $\rho^{5/2}$ are identical at the points $(\rho, \varphi_0^*, \psi_0)$ and $(\rho, \varphi_0, \psi_0)$. It follows from this and from the definition of the function $W(\mathbf{x})$ that $W(\mathbf{x}) = O(\rho^{5/2})$ in the half-line specified by the angles ψ_0 and φ_0 , that is, equalities (3.14) are satisfied along the given direction.

For the proof of Lemma 3 beside Lemma 4, we also use the enhanced maximum principle at the point Q for the domain S_+ . Since the point Q lies on the edge of a right dihedral angle, direct use of the enhanced Hopf maximum principle is impossible since one cannot inscribe a sphere in S_+ which only intersects ∂S_+ at the point Q . We therefore use another version of the enhanced maximum principle [11] which is suitable for this case. We will formulate it in a form sufficient to prove Lemma 3.

Assertion 3. Suppose $D^* \subset R^n$ is a domain bounded by a smooth surface and that Π is a hyperplane which passes through a joint $Q \in \partial D^*$ and is normal to the surface ∂D^* at the given point. Suppose D is the part of the domain D^* which lies on one side of Π . We will assume that the function $w(x)$, which is defined in the closure of the domain D , satisfies the following conditions

$$\Delta w(x) \leq 0, \quad w(x) \geq 0, \quad x \in D; \quad w(Q) = 0; \quad w(x') = 0, \quad x' \in \Pi \cap D^*$$

Suppose l is an arbitrary direction which passes into the domain D and is not tangential to the surface ∂D at the point Q . We will also assume that the derivatives $w_l(Q)$ and $w_{ll}(Q)$ exist, in which case one, of three possibilities holds:

- 1) $w_l(Q) > 0$; 2) $w_l(Q) = 0$, $w_{ll}(Q) > 0$;
- 3) if $w_l(Q) = 0$ and $w_{ll}(Q) = 0$, then $w(x) \equiv \text{const}$

The assertion of Lemma 3 immediately follows from Lemma 1 and 4 and the above mentioned version of the enhanced maximum principle. Actually, according to Lemma 1, the harmonic function $W(\mathbf{x})$ which has been constructed is non-negative. Furthermore, it satisfies conditions (3.8) and (3.14). It therefore follows from Assertion 3 that $W(\mathbf{x}) \equiv \text{const}$, which contradicts the assumption that $G_s \neq G_+$.

It is now possible to complete the proof of the main theorem. It follows from Lemmas 2 and 3 that, if $G_s \neq G_+$, then neither of the two possible versions of the limiting position can be realized. Consequently, in the limiting position $G_s = G_+$, that is, the domain G is symmetrical about the line T . Since the line T had an arbitrary direction, it follows from this that, for any direction, there is a line having the given direction about which the domain G is symmetrical. According to Assertion 1, this means that the domain G is a circle.

Remark. It has been proved in [8]† that constancy of the stress intensity factor is a necessary condition for the contour of the crack to be extremal. It follows from the theorem proved above that, in the case of a uniform load, this condition is also a sufficient condition.

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†See also GOL'DSHTEIN, R. V. and ENTOV, V. M., Some qualitative methods in Fracture mechanics. Preprint No. 76, Inst. Problem Mekhaniki Akad. Nauk SSSR, Moscow, 1976.